Fort Street
High School

Solutions

2024 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Approved calculators may be used
- A reference sheet is provided
- Marks may be deducted for careless or badly arranged work.
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks : 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Allow about 2 hours and 45 minutes for this section
- Write your student number on each answer booklet.
- Attempt Questions 11 – 16

Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

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1. Ler $\underset{\sim}{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

Which of the following is the value of $\underset{\sim}{a}\cdot\left(\underset{\sim}{a}-3\underset{\sim}{b}\right)$?

A. -7

B. 0

C. 3

D. 6

$\underset{\sim}{a}\cdot(\underset{\sim}{a}-3\underset{\sim}{b})$

$=\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-8\\5\\-3\end{pmatrix}$

$=-8+10-9$

$=-7$

2. Which of the following is an expression for $\displaystyle\int\frac{x^3-1}{\left(x^4-4x\right)^{\frac{2}{3}}}dx$?

A. $\dfrac{3}{4(x^3-4x)}+C$

B. $\dfrac{3}{4}(x^3-4x)+C$

C. $\dfrac{3}{4}\sqrt[3]{x^4-4x}+C$

D. $\dfrac{3}{4\sqrt[3]{x^4-4x}}+C$

$\frac{1}{4}\int\frac{4(x^3-1)}{(x^4-4x)^{\frac{2}{3}}}\,dx$

$=\frac{1}{4}\int(x^4-4x)^{-2/3}(4x^3-4).\,dx$

$=\frac{1}{4}\frac{(x^4-4x)^{\frac{1}{3}}}{\frac{1}{3}}+C$

$=\frac{3}{4}(x^4-4x)^{1/3}+C$

$=\frac{3}{4}\sqrt[3]{x^4-4x}+C$

3. What is the value of $\displaystyle\int_0^{\frac{\pi}{2}}x\sin x\ \ dx$?

A. $\dfrac{\pi}{4}$

B. $\dfrac{\pi}{2}$

C. π

D. 1

$\int_0^{\frac{\pi}{2}} x\sin x\,dx$

$=\left[-x\cos x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}1\times-\cos x\,dx$

$=\left[0-0\right]+\left[\sin x\right]_0^{\frac{\pi}{2}}$

$=1-0$

$=1$

4. In which quadrant does the complex number $2e^{\frac{-i5\pi}{12}}+2e^{\frac{i\pi}{12}}$ lie ?

A. I

B. II

C. III

D. IV

$2e^{-i\frac{5\pi}{12}} + 2e^{i\frac{\pi}{12}}$

$= 2e^{-i\frac{\pi}{6}}\left(e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}}\right)$

$= 2e^{-i\frac{\pi}{6}}\left(2\cos\frac{\pi}{4}\right)$

$= 4\times\frac{1}{\sqrt{2}}\, e^{-i\frac{\pi}{6}}$

$\therefore$ 4th Quadrant

5. For how many integer values of n, where $i^2=-1$, is $n^4+(n+i)^4$ an integer?

A. 4

B. 3

C. 2

D. 1

$n^4 + (n+i)^4$

$= n^4 + {}^4C_0 n^4 + {}^4C_1 n^3 i + {}^4C_2 n^2 i^2 + {}^4C_3 n i^3 + {}^4C_4 i^4$

${}^4C_1 n^3 - {}^4C_3 n = 0$

$4n^3 - 4n = 0$

$4n(n^2-1) = 0$

$n(n-1)(n+1) = 0$

$n=0$ or $n=1$ or $n=-1$

3 values of n $\boxed{B}$

6. The complex number $z=a+ib$, where $0<a<b$.

Which of the following best describes the complex number z^4?

A. $\text{Re}(z^4)<0$

B. $\text{Re}(z^4)\le 0$

C. $\text{Im}(z^4)<0$

D. $\text{Im}(z^4)\le 0$

$z = a+ib$, $a>0$, $b>0$, $a<b$

Since $a<b$

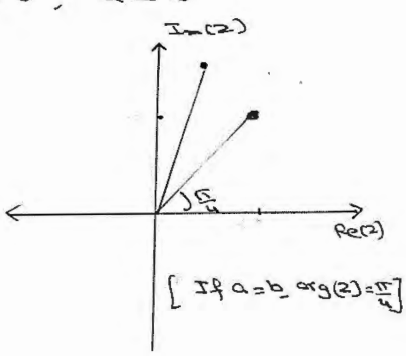
$\frac{\pi}{4} < \arg(z) < \frac{\pi}{2}$

$\arg(z^4) = 4\arg(z)$

$\pi < 4\arg(z) < 2\pi$

$\pi < \arg(z^4) < 2\pi$

$\therefore \text{Im}(z^4) < 0$



7. Consider the position vector of a particle.

$$\underset{\sim}{r}(t) = -3\sin(t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j} + t\underset{\sim}{k} \ .$$

Which of the following statements best describes the motion of the particle?

A. A spiral about the z axis in an anticlockwise direction.

B. A spiral about the z axis in a clockwise direction.

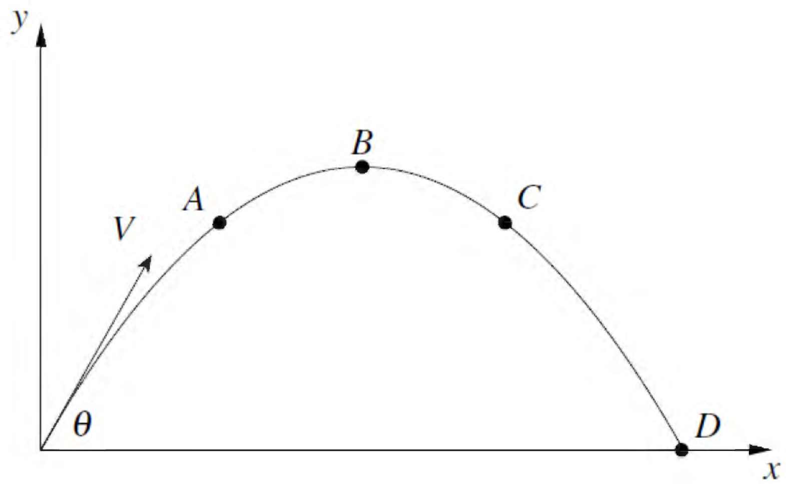
C. A spiral about the x axis in an anticlockwise direction.

D. A spiral about the x axis in a clockwise direction.

$x = -3\sin t, \quad y = 3\cos t, \quad z = t$

Spiral about z-axis in anticlockwise direction

8. A particle is projected from the origin, reaches a maximum height at point B and lands at point D. The acceleration of the particle is given by $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ and the velocity of the particle is $\underset{\sim}{v}(t)$.



For any initial velocity $V > 0$ and the angle of projection $0 < \theta < 90^\circ$, at which point on the trajectory of the particle are $\underset{\sim}{v}(t) \bullet \underset{\sim}{a}(t) < 0$ and $\underset{\sim}{r}(t) \bullet \underset{\sim}{v}(t) > 0$ always true?

A. A

B. B

C. C

D. D

9. A vehicle is moving horizontally on a frictionless surface in a resistive medium.

The resistive force is proportional to the square of the velocity of the vehicle.

The vehicle has a driving force that varies so that it is always half of the resistive force.

The initial speed of the particle is $5\ ms^{-1}$.

Which of the following is always true about the motion of the particle?

A. The velocity increases until it eventually comes to rest.

B. The velocity decreases until it eventually comes to rest.

C. The velocity increases until it eventually reaches its terminal velocity.

D. The velocity decreases until it eventually reaches $v = 2\ ms^{-1}$

$$ma = \frac{1}{2}kv^2 - kv^2$$

where k is the constant of proportionality

$$ma = -\frac{1}{2}kv^2$$

$$a = -\frac{1}{2}\frac{k}{m}v^2$$

$$\ddot{x} = -\frac{1}{2}\frac{k}{m}v^2$$

$$v\frac{dv}{dx} = -\frac{1}{2}\frac{k}{m}v^2$$

$$\frac{dv}{dx} = -\frac{1}{2}\frac{k}{m}v$$

$$\int_5^v \frac{1}{v}\,dv = -\frac{1}{2}\frac{k}{m}\int_0^x dx$$

$$\ln\frac{v}{5} = -\frac{1}{2}\frac{k}{m}x$$

$$\frac{v}{5} = e^{-\frac{kx}{2m}}$$

$$v = 5e^{-\frac{kx}{2m}}$$

As $x \to \infty$, $v \to 0$

10. The complex number z satisfies $|z+a| = a$,where a is a positive real number.

The point P represents the complex number $ka + a(k+1)i$,
where k is a positive real number.

The greatest distance that z can be at from the point P is $(3\sqrt{2}+1)a$.

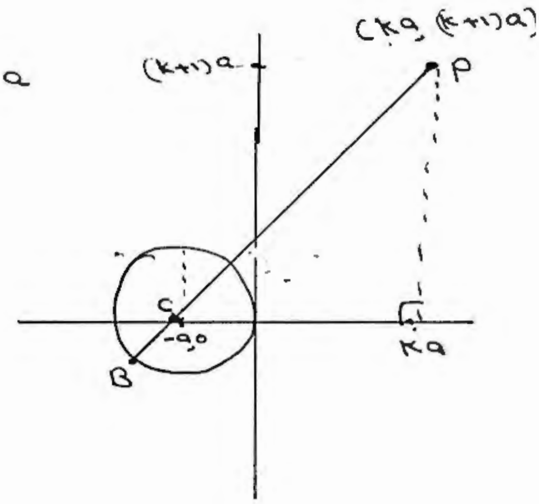
What is the value of k?

A. 1
B. 2
C. 3
D. 4

$$|z - (-a)| = a$$

Centre $(-a, 0)$, radius $= a$

$k > 0$



$$PC = \sqrt{(ka+a)^2 + ((k+1)a)^2}$$

$$= \sqrt{(k+1)^2a^2 + (k+1)^2a^2}$$

$$= \sqrt{2}(k+1)a$$

$$PB = PC + CB$$

$$= \sqrt{2}(k+1)a + a$$

$$= a((k+1)\sqrt{2} + 1)$$

$$a((k+1)\sqrt{2} + 1) = (3\sqrt{2}+1)a$$

$$(k+1)\sqrt{2} + 1 = 3\sqrt{2} + 1$$

$$(k+1)\sqrt{2} = 3\sqrt{2}$$

$$k+1 = 3$$

$$k = 2$$

$\boxed{B}$

End of Section I

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet.

(a) The position vectors of two points , A and B are given by $\overrightarrow{OA} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\overrightarrow{OB} = -\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$.

(i) Determine the exact distance between the points A and B. **1**

i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -3\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$|\overrightarrow{AB}| = \sqrt{(-3)^2 + (-3)^2 + 5^2}$

$= \sqrt{9 + 9 + 25}$

$= \sqrt{43}$ ✓

(ii) Show that there is no value of m such that $\overrightarrow{OC} = m\underset{\sim}{i} + 2\underset{\sim}{j} - m^2\underset{\sim}{k}$ is perpendicular to $\overrightarrow{OA}$. **2**

If $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{OA}$

then $\overrightarrow{OC} \cdot \overrightarrow{OA} = 0$

$(m\underset{\sim}{i} + 2\underset{\sim}{j} - m^2\underset{\sim}{k}) \cdot (2\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}) = 0$

$2m + 8 + 3m^2 = 0$

$3m^2 + 2m + 8 = 0$ ✓

$\Delta = 2^2 - 4 \times 3 \times 8$

$= 4 - 96$

$= -92 < 0$

$\therefore$ There is no value of m such that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{OA}$ ✓

(b) Solve $z^2-(2+6i)z=(5-2i)$. Give your answer in the form $x+iy$, **3**

where x and y are real.

$$z^2-(2+6i)z-(5-2i)=0$$

$$\Delta = \left(-(2+6i)\right)^2 - 4\times1\times(-(5-2i))$$

$$= 4(1+3i)^2 + 4(5-2i)$$

$$= 4(1-9+6i) + 4(5-2i)$$

$$= 4(-8+6i) + 4(5-2i)$$

$$= -32+24i+20-8i$$

$$= -12+16i$$

$$= 4(-3+4i)$$

$$\sqrt{\Delta} = \sqrt{4(-3+4i)} = 2\sqrt{-3+4i}$$

Let $(x+iy)^2 = -3+4i$

$x^2-y^2=-3$ ① $\qquad 2xy=4$ ②

$$(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$$

$$= 9+16$$

$$= 25$$

$x^2+y^2=5$ ③

① + ③ $\quad 2x^2 = 2$

$$x^2=1$$

$$x=\pm1$$

If $x=1$ $\qquad$ If $x=-1$

$y=2$ $\qquad\qquad y=-2$

$$\therefore \sqrt{-3+4i} = \pm(1+2i)$$

$$z = \frac{-(-(2+6i)) \pm 2(1+2i)}{2}$$

$$= (1+3i) \pm (1+2i)$$

$z = 1+3i+1+2i$ or $z = 1+3i-1-2i$

$= 2+5i$ $\qquad\qquad = i$

Alternative Solution:

$$z^2 - (2+6i)z - (5-2i) = 0$$

$$z^2 - (2+6i)z + (-5+2i) = 0$$

$$z^2 - 2(1+3i)z + (-5+2i) = 0$$

$$\Delta = (-2(1+3i))^2 - 4\times(-5+2i)$$

$$= 4(1-9+6i) + 20-8i$$

$$= 4(-8+6i) + 20-8i$$

$$= -32+24i+20-8i$$

$$= -12+16i$$

Let $(x+iy)^2 = -12+16i$

$x^2-y^2 = -12$ ① $\qquad 2xy = 16$ ②

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$= 144 + 256$$

$$= 400$$

$x^2+y^2 = 20$ — ③

from ① and ③ $\quad 2x^2 = 8$

$$x^2 = 4$$

$$x = \pm 2$$

$x = 2, \quad y = \frac{16}{2\times 2} = 4$

$x = -2, \quad y = \frac{16}{2\times -2} = -4$

$\therefore \quad x+iy = 2+4i \quad$ or $\quad x+iy = -2-4i$

$\therefore \quad \sqrt{-12+16i} = 2+4i \quad$ or $\quad \sqrt{-12+16i} = -2-4i$

$$z = \frac{(2+6i) \pm (2+4i)}{2}$$

$z = \frac{2+6i+2+4i}{2} \quad$ or $\quad z = \frac{2+6i-2-4i}{2}$

$z = \frac{4+10i}{2} \quad$ or $\quad z = \frac{2i}{2}$

$\boxed{z = 2+5i} \quad$ or $\quad \boxed{z = i}$ $\quad$ $\boxed{3}$

(c) Find:

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx \qquad \textbf{3}$$

$$I = \int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$$

$$= \int \frac{\cos^2 x}{\sqrt{\sin x}} \cos x \, dx$$

$$= \int \frac{1-\sin^2 x}{\sqrt{\sin x}} \cos x \, dx \quad \checkmark$$

$$= \int \left((\sin x)^{-\frac{1}{2}} - (\sin x)^{\frac{3}{2}} \right) \cos x \, dx$$

$$= \frac{(\sin x)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{(\sin x)^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + C$$

$\checkmark \qquad \checkmark$

(d) A particle moves along the $x-$axis with velocity v and acceleration a according to the equation $a = v^3 + 4v$. The particle starts at the origin with velocity $2\ cm/s$.

Find the expression for the displacement of the particle x, in terms of v. **3**

$a = v^3 + 4v$

$t = 0, \quad x = 0, \quad v = 2\ cm/s$

$a = v^3 + 4v$

$v\dfrac{dv}{dx} = v^3 + 4v$

$\dfrac{dv}{dx} = v^2 + 4$ ✓

$\displaystyle\int_2^v \frac{dv}{v^2+4} = \int_0^x dx$

$\displaystyle\int_2^v \frac{dv}{v^2+2^2} = \int_0^x dx$

$\dfrac{1}{2}\left[\tan^{-1}\dfrac{v}{2}\right]_2^v = \left[x\right]_0^x$ ✓

$\dfrac{1}{2}\left[\tan^{-1}\dfrac{v}{2} - \tan^{-1}\dfrac{2}{2}\right] = \left[x - 0\right]$

$\dfrac{1}{2}\left[\tan^{-1}\dfrac{v}{2} - \dfrac{\pi}{4}\right] = x$

$x = \dfrac{1}{2}\tan^{-1}\dfrac{v}{2} - \dfrac{\pi}{8}$ ✓

(e) The complex numbers z_1 and z_2 are given by $z_1 = 3 - i$ and $z_2 = 1 - 2i$.

Determine the possible value/s of the real constant k if $\left| \frac{z_1}{z_2} + k \right| = \sqrt{k+2}$. **3**

$$\frac{z_1}{z_2} = \frac{3-i}{1-2i}$$

$$= \frac{3-i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{3+2+6i-i}{1+4}$$

$$= \frac{5+5i}{5}$$

$$= 1+i$$

$$\left| \frac{z_1}{z_2} + k \right| = \sqrt{k+2}$$

$$\left| (1+i) + k \right| = \sqrt{k+2}$$

$$\left| (1+k) + i \right| = \sqrt{k+2}$$

$$\sqrt{(1+k)^2 + 1^2} = \sqrt{k+2}$$

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$$(1+k)^2 + 1 = k+2$$

$$k^2 + 2k + 2 = k+2$$

$$k^2 + 2k = k$$

$$k^2 + k = 0$$

$$k(k+1) = 0$$

$$\boxed{k=0} \text{ or } \boxed{k=-1}$$

End of Question 11

Question 12 (15 marks) Use a separate writing booklet.

(a) (i) Find A, B, and C such that **2**

$$\frac{1-2x}{(x+2)(x^2+1)}=\frac{A}{(x+2)}+\frac{Bx+C}{(x^2+1)}$$

(i) $\dfrac{1-2x}{(x+2)(x^2+1)} = \dfrac{A}{(x+2)} + \dfrac{Bx+C}{(x^2+1)}$

$1-2x = A(x^2+1) + (Bx+C)(x+2)$

Let $x=-2$

$1-2\times-2 = A((-2)^2+1)$

$5 = A\times 5$

$\boxed{A=1}$

Comparing coefficients of x^2

$0 = A+B$

$B = -A$

$\boxed{B=-1}$

Let $x=0$

$1-2\times 0 = A(0+1) + (B\times 0+C)(0+2)$

$1 = A+2C$

$1 = 1+2C$

$\boxed{C=0}$

✓ any two correct

✓✓ all A, B, C correct

(ii) Hence find $\int \frac{1-2x}{(x+2)(x^2+1)}dx$ **2**

$$I = \int \frac{1-2x}{(x+2)(x^2+1)} dx$$

$$= \int \left(\frac{1}{(x+2)} + \frac{-x+0}{x^2+1} \right) dx$$

$$= \int \frac{1}{x+2} dx - \frac{1}{2} \int \frac{2x}{x^2+1} dx$$

$$= \ln|x+2| - \frac{1}{2} \ln|x^2+1| + C$$

(b) Consider the lines **3**

$l_1: \quad x = y = z$

$l_2: \quad \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, where t is a parameter.

Show that the lines are skew.

(b) $l_1: \quad x = y = z$

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$$

$l_2: \quad \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ where t is a parameter

$$= \begin{pmatrix} t \\ 3-2t \\ 2+t \end{pmatrix}$$

The lines are not parallel

as $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ where μ is a scalar ✓

or $\frac{1}{1} \neq \frac{1}{-2}$

For line l_1:

$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ where λ is a scalar

$$= \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}$$

If lines l_1 and l_2 intersect

$\lambda = t$ ①, $\lambda = 3-2t$ ②, $\lambda = 2+t$ ③ ✓

Using ① and ② $\quad \lambda = 3 - 2\lambda$

$3\lambda = 3$

$\lambda = 1$

$\therefore \lambda = t = 1$

Substitute λ and t in ③

L.H.S. $= \lambda = 1$

R.H.S. $= 2+t = 2+1 = 3$

L.H.S. $\neq$ R.H.S.

$\therefore$ The lines l_1 and l_2 don't intersect.

Since l_1 and l_2 are neither parallel nor intersect.

$\therefore$ The lines l_1 and l_2 are skew.

✓

(c) The polynomial $P(z) = z^4 - 8z^3 + pz^2 + qz - 80$ has root $3+i$, where p and q are real numbers.

(i) Find all the roots of $P(z)$. **3**

$P(z) = z^4 - 8z^3 + pz^2 + qz - 80$

Since all the Coefficients are real and $3+i$ is one root of $P(z)$, $3-i$ is also a root of $P(z)$. ✓

Let α and β are one roots of $P(z)$

Sum of roots $= 8$

$(3+i)+(3-i) + \alpha + \beta = 8$

$6 + \alpha + \beta = 8$

$\boxed{\alpha + \beta = 2}$ ①

Product of Roots $= -80$

$(3+i)(3-i) \times \alpha \times \beta = -80$

$(9+1)\alpha\beta = -80$

$\boxed{\alpha\beta = -8}$ ② ✓

Substitute $\alpha = 2 - \beta$ from ① into ②

$(2-\beta)\beta = -8$

$2\beta - \beta^2 = -8$

$\beta^2 - 2\beta - 8 = 0$

$(\beta - 4)(\beta + 2) = 0$

$\beta = 4$ or $\beta = -2$

If $\beta = 4$, $\alpha = 2 - 4 = -2$

$\beta = -2$, $\alpha = 2-(-2) = 4$

$\therefore$ The roots of $P(z)$ are $3+i$, $3-i$, -2 and 4 ✓

(ii) Write $P(z)$ as a product of two real quadratic factors. **1**

$P(z) = (z^2 - 6z + 9)(z+2)(z-4)$

$P(z) = (z^2 - 6z + 10)(z^2 - 2z - 8)$

(d) Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{3}} \frac{1}{4+5\cos x}dx$. **4**

$$I = \int_0^{\pi/3} \frac{1}{4+5\cos x} dx$$

Let $t = \tan\frac{x}{2}$

$dt = \left(\sec^2\frac{x}{2}\right) \times \frac{1}{2} dx$

$= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) dx$

$= \frac{1}{2}(1+t^2) dx$

$dx = \frac{2dt}{1+t^2}$

When $x = 0$, $t = 0$

$x = \frac{\pi}{3}$ $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\cos x = \frac{1-t^2}{1+t^2}$ ✓

$$\therefore I = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{\cancel{(1+t^2)}}{4(1+t^2)+5(1-t^2)} \times \frac{2dt}{\cancel{(1+t^2)}}$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{4+4t^2+5-5t^2} dt$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{-t^2+9}\,dt$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{9-t^2}\,dt$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(3+t)(3-t)}\,dt$$

Let $\dfrac{1}{(3+t)(3-t)} = \dfrac{A}{3+t} + \dfrac{B}{3-t}$

$1 = A(3-t) + B(3+t)$

Let $t=3$

$1 = A\times 0 + B\times 6 \Rightarrow \boxed{B = \frac{1}{6}}$

Let $t=-3$

$1 = A\times 6 + B\times 0 \Rightarrow \boxed{A = \frac{1}{6}}$

$$\therefore I = 2\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{1}{6(3+t)} + \frac{1}{6(3-t)}\right) dt$$

$$= \frac{2}{6}\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{1}{3+t} + \frac{1}{3-t}\right) dt$$

$$= \frac{1}{3}\Big[\ln|3+t| - \ln|3-t|\Big]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{3}\left[\ln\left|\frac{3+t}{3-t}\right|\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{3}\left[\ln\left|\frac{3+\frac{1}{\sqrt{3}}}{3-\frac{1}{\sqrt{3}}}\right| - \ln\left|\frac{3+0}{3-0}\right|\right]$$

$$= \frac{1}{3}\left[\ln\left|\frac{3\sqrt{3}+1}{3\sqrt{3}-1}\right| - \ln 1\right]$$

$$= \frac{1}{3}\left[\ln\left|\frac{3\sqrt{3}+1}{3\sqrt{3}-1}\right| - 0\right]$$

$$= \frac{1}{3}\ln\left(\frac{3\sqrt{3}+1}{3\sqrt{3}-1}\right)$$

Question 13 (14 marks) Use a separate writing booklet.

(a) Find $\displaystyle\int \frac{(2\tan\theta+3)\sec^2\theta}{\sec^2\theta+\tan\theta}\,d\theta$ **3**

$$I = \int \frac{(2\tan\theta+3)\sec^2\theta}{\sec^2\theta+\tan\theta}\,d\theta$$

Let $\tan\theta = u$ $\qquad \sec^2\theta = 1+\tan^2\theta$

$\sec^2\theta\, d\theta = du$ $\qquad = 1+u^2$

$$I = \int \frac{(2u+3)}{1+u^2+u}\,du \quad ✓$$

$$= \int \frac{(2u+1)}{u^2+u+1}\,du + \int \frac{2}{u^2+u+1}\,du$$

$$= \ln|u^2+u+1| + 2\int \frac{du}{u^2+u+\frac{1}{4}-\frac{1}{4}+1}$$

$$= \ln|u^2+u+1| + 2\int \frac{du}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \ln|u^2+u+1| + 2\times\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \ln|u^2+u+1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)+C$$

$$= \ln|\tan^2\theta+\tan\theta+1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta+1}{\sqrt{3}}\right)+C$$

✓ ✓

(b) A particle of mass 1 *kg* is projected from the origin with initial speed V m/s at an angle α to the horizontal plane.
The parametric equations of motion are given by

$$\ddot{x} = -5\dot{x} \qquad \text{and} \qquad \ddot{y} = -10 - 5\dot{y}$$

The position vector of the particle , at any time t seconds after the particle is projected , is $\vec{r}(t)$ and the velocity vector is $\vec{v}(t)$.

(i) Show that $\vec{v}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha + 2)e^{-5t} - 2 \end{pmatrix}$ **3**

$$\ddot{x} = -5\dot{x}$$

$$\frac{d\dot{x}}{dt} = -5\dot{x}$$

$$\int_{V\cos\alpha}^{\dot{x}} \frac{1}{\dot{x}}\, d\dot{x} = -5\int_0^t dt$$

$$\left[\ln|\dot{x}|\right]_{V\cos\alpha}^{\dot{x}} = -5\left[t\right]_0^t$$

$$\ln|\dot{x}| - \ln|V\cos\alpha| = -5[t-0]$$

$$\ln\left|\frac{\dot{x}}{V\cos\alpha}\right| = -5t$$

$$\frac{\dot{x}}{V\cos\alpha} = e^{-5t}$$

$$\boxed{\dot{x} = Ve^{-5t}\cos\alpha} \quad ① \quad ✓$$

$$\ddot{y} = -10 - 5\dot{y}$$

$$\frac{d\dot{y}}{dt} = -5(2+\dot{y})$$

$$\int_{V\sin\alpha}^{\dot{y}} \frac{1}{2+\dot{y}}\, d\dot{y} = -5\int_0^t dt$$

$$\left[\ln|2+\dot{y}|\right]_{V\sin\alpha}^{\dot{y}} = -5\left[t\right]_0^t \quad ✓$$

$$\ln|2+\dot{y}| - \ln|2+V\sin\alpha| = -5[t-0]$$

$$\ln\left|\frac{2+\dot{y}}{2+V\sin\alpha}\right| = -5t$$

$$\frac{2+\dot{y}}{2+V\sin\alpha} = e^{-5t}$$

$$2+\dot{y} = (2+V\sin\alpha)e^{-5t}$$

$$\boxed{\dot{y} = (2+V\sin\alpha)e^{-5t} - 2} \quad (2)$$

$$\vec{v}(t) = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\vec{V}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha+2)e^{-5t} - 2 \end{pmatrix}$$ using (1) and (2)

(ii) Given that $\vec{v}(1) = \begin{pmatrix} 250e^{-5} \\ \left(250\sqrt{3}+2\right)e^{-5}-2 \end{pmatrix}$, **2**

find the initial speed V and the angle of projection α.

$t = 1$

$$Ve^{-5\times1}\cos\alpha = 250e^{-5}$$

$$\therefore \boxed{V\cos\alpha = 250} \quad (3)$$

$$(V\sin\alpha+2)e^{-5\times1} - 2 = (250\sqrt{3}+2)e^{-5} - 2$$

$$V\sin\alpha + 2 = 250\sqrt{3} + 2$$

$$\boxed{V\sin\alpha = 250\sqrt{3}} \quad (4)$$

$(4) \div (3)$ $$\frac{\cancel{V}\sin\alpha}{\cancel{V}\cos\alpha} = \frac{250\sqrt{3}}{250}$$

$$\tan\alpha = \sqrt{3}$$

$$\boxed{\alpha = 60^\circ}$$

Substitute $\alpha = 60^\circ$ into (1)

$$V\cos 60^\circ = 250$$

$$V \times \frac{1}{2} = 250$$

$$V = 500 \text{ m/s}$$

(iii) Show that the ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is: **2**

$$\left(1+125\sqrt{3}\right):1$$

Horizontal velocity at the origin $= V\cos\alpha$

$= 500\sqrt{3}$

At maximum height :-

$\dot{y} = 0$

$(V\sin\alpha + 2)e^{-5t} - 2 = 0$ from (i)

$\left(500 \times \frac{\sqrt{3}}{2} + 2\right)e^{-5t} = 2$

$(250\sqrt{3}+2)e^{-5t} = 2$

$e^{-5t} = \frac{2}{250\sqrt{3}+2}$

$e^{-5t} = \frac{1}{125\sqrt{3}+1}$

Horizontal velocity at maximum height

$= V \times \frac{1}{125\sqrt{3}+1} \times \cos\alpha$

$= 500 \times \frac{1}{125\sqrt{3}+1} \times \sqrt{3}$

$= \frac{500\sqrt{3}}{125\sqrt{3}+1}$

$\therefore$ The Ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is :

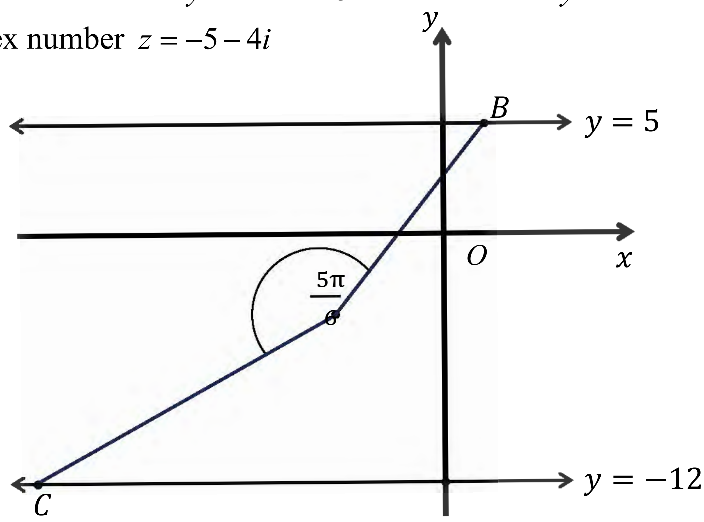
$500\sqrt{3} : \frac{500\sqrt{3}}{125\sqrt{3}+1}$

$\boxed{125\sqrt{3}+1 : 1}$

Hence shown

(c) In the Argand diagram below, B lies on the line $y = 5$ and C lies on the line $y = -12$. **4**

The point A represents the complex number $z = -5 - 4i$



NOT TO SCALE

Given $AC = 2AB$ and $\angle BAC = \dfrac{5\pi}{6}$.

Find the exact complex number that represents the point C.

$$\vec{AC} = 2\vec{AB}\ \text{Cis}\ \frac{5\pi}{6}$$

$$(\vec{OC} - \vec{OA}) = 2(\vec{OB} - \vec{OA})\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$(\vec{OC} - \vec{OA}) = 2(\vec{OB} - \vec{OA})\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) \quad ✓ \quad ①$$

Let Point C represents the Complex number

$z_1 = a - 12i$

and Point B represents the Complex number

$z_2 = b + 5i$

From ①

✓

$$((a - 12i) - (-5 - 4i)) = \cancel{2}((b + 5i) - (-5 - 4i))\left(\frac{-\sqrt{3} + i}{\cancel{2}}\right)$$

$$(a + 5) + (-12 + 4)i = ((b + 5) + (5 + 4)i)(-\sqrt{3} + i)$$

$$(a + 5) - 8i = ((b + 5) + 9i)(-\sqrt{3} + i)$$

$$(a + 5) - 8i = (-\sqrt{3}(b + 5) - 9) + i(b + 5 - 9\sqrt{3})$$

$a + 5 = -\sqrt{3}(b + 5) - 9$, $b + 5 - 9\sqrt{3} = -8$

$\therefore b = -13 - 9\sqrt{3}$

✓

$a = -\sqrt{3}(b + 5) - 14$

$a = -\sqrt{3}(-8 + 9\sqrt{3}) - 14$

$= 8\sqrt{3} - 27 - 14$

$= 8\sqrt{3} - 41$ $\boxed{4}$

$\therefore$ The Complex number which represents C

is $z_1 = (8\sqrt{3} - 41) - 12i$ ✓

Question 14. (16 marks) Use a separate writing booklet.

(a) Given that $z = e^{\frac{\pi}{11}i}$, show that

(i) Show that $z + z^3 + z^5 + z^7 + z^9 = \dfrac{1}{1-z}$ **2**

(i) $z + z^3 + z^5 + z^7 + z^9$

$= z\dfrac{((z^2)^5 - 1)}{z^2 - 1}$

$= z\dfrac{(z^{10} - 1)}{z^2 - 1}$

$= \dfrac{z^{11} - z}{z^2 - 1}$ ✓

$z^{11} = \left(e^{\frac{\pi}{11}i}\right)^{11} = e^{\pi i} = \cos\pi + i\sin\pi = -1$

$\therefore\ z + z^3 + z^5 + z^7 + z^9$

$= \dfrac{-1 - z}{z^2 - 1}$

$= -\dfrac{(1+z)}{(z^2-1)}$

$= -\dfrac{\cancel{(1+z)}}{\cancel{(z+1)}(z-1)}$

$= \dfrac{-1}{z-1}$

$= \dfrac{1}{1-z}$ ✓

Marker's Comment:-
Most of the students couldn't recognise G.P. Very few students received one or two marks.

[?]

(ii) Hence show that $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$ **3**

$$\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$$

$$= \text{Re}\,(z+z^3+z^5+z^7+z^9)$$

$$= \text{Re}\left(\frac{1}{1-z}\right) \quad ✓$$

$$\frac{1}{1-z} = \frac{1}{1-\left(\cos\frac{\pi}{11}+i\sin\frac{\pi}{11}\right)}$$

$$= \frac{1}{\left(1-\cos\frac{\pi}{11}\right)-i\sin\frac{\pi}{11}} \times \frac{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}$$

$$= \frac{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}{\left(1-\cos\frac{\pi}{11}\right)^2+\left(\sin\frac{\pi}{11}\right)^2} \quad ✓$$

$$= \frac{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}{1+\cos^2\frac{\pi}{11}-2\cos\frac{\pi}{11}+\sin^2\frac{\pi}{11}}$$

$$= \frac{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}{1+\left(\cos^2\frac{\pi}{11}+\sin^2\frac{\pi}{11}\right)-2\cos\frac{\pi}{11}}$$

$$= \frac{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}{1+1-2\cos\frac{\pi}{11}}$$

$$= \frac{\left(1-\cos\frac{\pi}{11}\right)+i\sin\frac{\pi}{11}}{2\left(1-\cos\frac{\pi}{11}\right)}$$

$$\text{Re}\left(\frac{1}{1-z}\right) = \frac{\cancel{\left(1-\cos\frac{\pi}{11}\right)}}{2\cancel{\left(1-\cos\frac{\pi}{11}\right)}}$$

$$= \frac{1}{2}$$

$$\therefore \cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$$

Hence shown ✓ $\boxed{3}$

Alternative Solution 1:

Q 14. (i)

(ii) $\frac{1}{1-z} = \frac{1}{(1-z)} \times \frac{(1-\bar{z})}{(1-\bar{z})}$

$= \frac{1-\bar{z}}{1-z-\bar{z}+z\bar{z}}$

$= \frac{1-\bar{z}}{1-(z+\bar{z})+|z|^2}$

$= \frac{1-\bar{z}}{1-2Re(z)+|z|^2}$

$= \frac{1-\bar{z}}{1-2\cos\frac{\pi}{11}+1} \qquad \because |z|^2=1$

$= \frac{1-(\cos\frac{\pi}{11}-i\sin\frac{\pi}{11})}{2-2\cos\frac{\pi}{11}}$

$= \frac{(1-\cos\frac{\pi}{11})+i\sin\frac{\pi}{11}}{2(1-\cos\frac{\pi}{11})}$

$Re\left(\frac{1}{1-z}\right) = \frac{1-\cos\frac{\pi}{11}}{2(1-\cos\frac{\pi}{11})}$

$= \frac{1}{2} \qquad (1)$

$Re(z+z^3+z^5+z^7+z^9)$

$= \cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11} \qquad (2)$

$\therefore$ Using a (i) and from (1) and (2)

$\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11} = \frac{1}{2}$

Alternative Solution 2:

$$\frac{1}{1-z} = \frac{1}{1-e^{\frac{\pi}{11}i}}$$

$$= \frac{1}{e^{\frac{\pi}{22}i}\left(e^{-\frac{\pi}{22}i} - e^{\frac{\pi}{22}i}\right)}$$

$$= \frac{e^{-\frac{\pi}{22}i}}{-2i\sin\frac{\pi}{22}}$$

$$= \frac{\cos\frac{\pi}{22} - i\sin\frac{\pi}{22}}{-2i\sin\frac{\pi}{22}}$$

$$= \frac{\cos\frac{\pi}{22}}{-2i\sin\frac{\pi}{22}} \times \frac{i}{i} + \frac{1}{2}$$

$$= \frac{\cos\frac{\pi}{22}}{2\sin\frac{\pi}{22}}\, i + \frac{1}{2}$$

$$\operatro{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2} \qquad (1)$$

$$\operatorname{Re}\left(z + z^3 + z^5 + z^7 + z^9\right)$$

$$= \cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} \qquad (2)$$

$\therefore$ From (1) and (2)

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$$

Alternative Solution 3;

$$z + z^3 + z^5 + z^7 + z^9 = \frac{1}{1-z}$$

$$\Rightarrow (1-z)(z + z^3 + z^5 + z^7 + z^9) = 1$$

$$z - z^2 + z^3 - z^4 + z^5 - z^6 + z^7 - z^8 + z^9 - z^{10} = 1 \checkmark$$

$$(z - z^{10}) + (-z^2 + z^9) + (z^3 - z^8) + (-z^4 + z^7)$$
$$+ (z^5 - z^6) = 1 \qquad (1)$$

Now $z^{11} = -1$

$$z^{10} \times z = -1$$

$$z^{10} = \frac{-1}{z}$$

$$= \frac{-1}{e^{\frac{\pi}{11}i}} = -e^{-\frac{\pi}{11}i}$$

$$z^2 = -\frac{1}{z^9} = -e^{-\frac{9\pi}{11}i}$$

$$z^8 = -\frac{1}{z^3} = -e^{-\frac{3\pi}{11}i}$$

$$z^4 = -\frac{1}{z^7} = -e^{-\frac{7\pi}{11}i}, \quad z^6 = -\frac{1}{z^5} = -e^{-\frac{5\pi}{11}i}$$

$\therefore$ (1) become

$$\left(e^{\frac{\pi}{11}i} + e^{-\frac{\pi}{11}i}\right) + \left(e^{-\frac{9\pi}{11}i} + e^{\frac{9\pi}{11}i}\right)$$
$$+ \left(e^{\frac{3\pi}{11}i} + e^{-\frac{3\pi}{11}i}\right) + \left(-e^{-\frac{7\pi}{11}i} + e^{\frac{7\pi}{11}i}\right)$$
$$+ \left(e^{\frac{5\pi}{11}i} - e^{-\frac{5\pi}{11}i}\right) = 1$$

$$2\cos\frac{\pi}{11} + 2\cos\frac{9\pi}{11} + 2\cos\frac{3\pi}{11} + 2\cos\frac{7\pi}{11}$$
$$+ 2\cos\frac{5\pi}{11} = 1$$

$$\left[\because e^{i\theta} + e^{-i\theta} = 2\cos\theta\right]$$

$$\Rightarrow \cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$$

Hence Shown

(b) (i) Use the substitution $u = \dfrac{1}{x}$ to show that $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = 0$. **2**

$u = \frac{1}{x}$ $\qquad$ $x = \frac{1}{\sqrt{3}}$, $u = \sqrt{3}$

$x = \frac{1}{u}$ $\qquad$ $x = \sqrt{3}$, $u = \frac{1}{\sqrt{3}}$

$dx = -\frac{1}{u^2}\,du$

$$I = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\ln\left(\frac{1}{u}\right)}{1+\left(\frac{1}{u}\right)^2} \times -\frac{1}{u^2}\,du$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} -\frac{\ln u}{u^2+1} \times \cancel{u^2} \times \frac{1}{\cancel{u^2}}\,du$$

$$= -\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln u}{u^2+1}\,du$$

$$= -I$$

$I + I = 0$

$2I = 0$

$I = 0$

$$\therefore \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = 0$$

Marker's Comment.
About 40 - 50 % Students got it correct. Students struggled in using properties of definite integrals.

$\boxed{2}$

(ii) Hence show that $\displaystyle\int_1^3 \frac{\ln x}{3+x^2}\,dx = \frac{\pi\sqrt{3}\ln 3}{36}$. **3**

4 (b)

(ii) $I = \int_1^3 \frac{\ln x}{3+x^2}\,dx$

Let $u = \frac{\sqrt{3}}{x}$ $\qquad x = 1,\ u = \sqrt{3}$

$x = \frac{\sqrt{3}}{u}$ $\qquad x = 3,\ u = \frac{1}{\sqrt{3}}$

$dx = -\frac{\sqrt{3}}{u^2}\,du$

$$I = \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\ln \frac{\sqrt{3}}{u}}{3+\left(\frac{\sqrt{3}}{u}\right)^2} \times -\frac{\sqrt{3}}{u^2}\,du \quad \checkmark$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln\sqrt{3} - \ln u}{3(u^2+1)} \times \cancel{u^2} \times \frac{\sqrt{3}}{\cancel{u^2}}\,du$$

$$= \frac{1}{\sqrt{3}}\left[\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln\sqrt{3}}{u^2+1}\,du - \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln u}{u^2+1}\,du\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln 3}{u^2+1}\,du - 0\right] \quad \checkmark \text{ Using (i)}$$

$$= \frac{1}{2\sqrt{3}}\ln 3 \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{u^2+1}\,du$$

$$= \frac{\ln 3}{2\sqrt{3}}\Big[\tan^{-1} u\Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \frac{\ln 3}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$= \frac{\ln 3}{2\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= \frac{\ln 3}{2\sqrt{3}} \times \frac{\pi}{6}$$

$$= \frac{\pi \ln 3}{12\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}\ln 3}{36} \quad \checkmark$$

Alternative Solution:

let $u = \frac{x}{\sqrt{3}}$

$du = \frac{1}{\sqrt{3}}dx$ $\qquad x = 1, \; u = \frac{1}{\sqrt{3}}$

$dx = \sqrt{3}\, du$ $\qquad x = 3, \; u = \sqrt{3}$

$$\therefore I = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln \sqrt{3}u}{3 + (\sqrt{3}u)^2} \times \sqrt{3}\, du$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{3} \frac{\ln\sqrt{3} + \ln u}{(1+u^2)} \times \sqrt{3}\, du$$

$$= \frac{1}{\sqrt{3}}\left[\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln 3}{1+u^2}\, du + \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln u}{1+u^2}\, du\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\ln 3}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+u^2}\, du + 0\right] \quad \text{from } i$$

$$= \frac{\ln 3}{2\sqrt{3}}\Big[\tan^{-1}u\Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}\ln 3\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$= \frac{1}{2\sqrt{3}}\ln 3\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= \frac{1}{2\sqrt{3}}\ln 3 \times \frac{\pi}{6}$$

$$= \frac{\pi \ln 3}{2 \times 6 \times \sqrt{3}} = \frac{\pi \ln 3}{12\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\pi \ln 3}{36}$$

(c) (i) A complex number z satisfies both $|z-1| \le |z-i|$ and $|z-2-2i| \le 1$. **3**

Sketch on an Argand diagram, the region which contains the point P representing z.

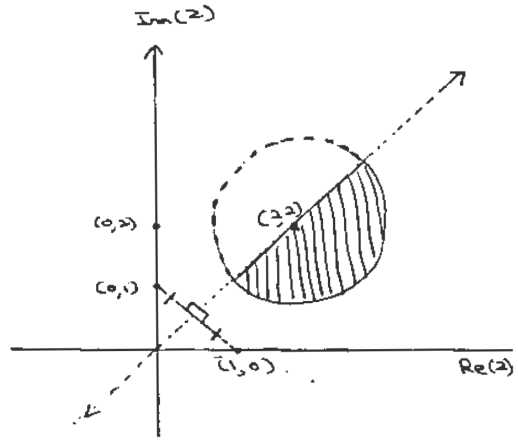
3)14 (c) (i) $|z-2-2i| = 1$

$|z-(2+2i)| = 1$

✓ For $|z-1| = |z-i|$

✓ For $|z-(2-2i)| = 1$

✓ For Correct Shaded region



(ii) Point $Q(w)$ lies on the boundary of the region obtained in part (i) **3**

and also satisfies $\arg(w-1) = \dfrac{\pi}{4}$.

Find all possible complex numbers w in the form of $x+iy$ where x and y are real.

Let $w = x+iy$

$(w-1) = (x-1)+iy$

$\arg(w-1) = \frac{\pi}{4}$

$\frac{y}{x-1} = 1$

$y = x-1$ ✓

Q lies on the boundary of $|z-(2+2i)| = 1$

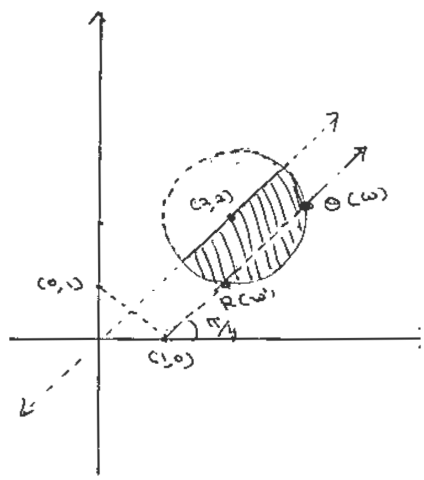
$|(x+iy)-(2+2i)| = 1$

$|(x-2)+i(y-2)| = 1$

As $y = x-1$

$|(x-2)+i(x-1-2)| = 1$

$|(x-2)+i(x-3)| = 1$ ✓



$(x-2)^2 + (x-3)^2 = 1$

$x^2 - 4x + 4 + x^2 - 6x + 9 = 1$

$2x^2 - 10x + 12 = 0$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$x = 3$ or $x = 2$

$y = 3 - 1$ $\quad$ $y = 2 - 1$

$= 2$ $\quad$ $= 1$

$\therefore \omega = 3 + 2i$ or $2 + i$ ✓

End of Question 14

Question 15. (15 marks) Use a separate writing booklet.

(a) A particle of mass $m\ kg$ is moving along the x *axis* under the action of the resisting force given by:

$$m\left(pv+v^2\right),$$

where v is its velocity of particle after t seconds and p is a positive constant.

Initially, the particle is at $x=\ln 2$ and is travelling with velocity p m/s.

(i) Show that the displacement x of the particle is given by **2**

$$x=\ln\left(\frac{4p}{p+v}\right).$$

$$ma = -m(pv+v^2)$$

$$a = -(pv+v^2)$$

$$v\frac{dv}{dx} = -(pv+v^2)$$

$$\frac{dv}{dx} = -(p+v)$$

$$\int_p^v \frac{dv}{p+v} = -\int_{\ln 2}^{x} dx \quad \checkmark$$

$$\left[\ln|p+v|\right]_p^v = -\left[x\right]_{\ln 2}^{x}$$

$$\ln|p+v| - \ln|2p| = -x + \ln 2$$

$$x = \ln 2 - \ln p+v + \ln 2p$$

$$x = \ln\left(\frac{2\times 2p}{p+v}\right)$$

$$x = \ln\left(\frac{4p}{p+v}\right) \quad \checkmark$$

(ii) Show that: $t = \frac{1}{p}\ln\left(\frac{p+v}{2v}\right)$. **3**

$$ma = -m(pv + v^2)$$

$$a = -(pv + v^2)$$

$$\frac{dv}{dt} = -(pv + v^2)$$

$$\int_p^v \frac{dv}{pv + v^2} = -\int_0^t dt$$

$$\int_p^v \frac{dv}{v(p+v)} = -\int_0^t dt \quad *$$

$$\frac{1}{v(p+v)} = \frac{A}{v} + \frac{B}{p+v}$$

$$1 = A(p+v) + Bv$$

Let $v = 0$

$$1 = A \times p \Rightarrow \boxed{A = \frac{1}{p}}$$

Let $v = -p$

$$1 = B \times -p \Rightarrow \boxed{B = -\frac{1}{p}}$$

$*$ becomes

$$\int_p^v \left(\frac{1}{pv} - \frac{1}{p(p+v)}\right) dv = -\int_0^t dt$$

$$\frac{1}{p}\int_p^v \left(\frac{1}{v} - \frac{1}{p+v}\right) dv = -\int_0^t dt$$

$$\frac{1}{p}\left[\ln|v| - \ln|p+v|\right]_p^v = -\left[t\right]_0^t$$

$$\frac{1}{p}\left[\ln\left|\frac{v}{p+v}\right|\right]_p^v = -\left[t - 0\right]$$

$$\frac{1}{p}\left[\ln\left|\frac{v}{p+v}\right| - \ln\left(\frac{1}{2}\right)\right] = -t$$

$$\frac{1}{p}\left[\ln\left|\frac{2v}{p+v}\right|\right] = -t$$

$$t = -\frac{1}{p}\ln\left|\frac{2v}{p+v}\right| = \frac{1}{p}\ln\left|\frac{v+p}{2v}\right|$$

(iii) It took the particle $\frac{1}{2}\ln 2$ seconds to reach the point where $x = \ln 3$ meters. **2**

Find the value of p.

Using (i)

$\ln 3 = \ln\left(\frac{4p}{p+v}\right)$

$3 = \frac{4p}{p+v}$

$3p + 3v = 4p$

$3v = p$

$v = \frac{p}{3}$

Substitute $v = \frac{p}{3}$ and $t = \frac{1}{2}\ln 2$ into

$t = \frac{1}{p}\ln\left(\frac{p+v}{2v}\right)$ (using (ii))

$\frac{1}{2}\ln 2 = \frac{1}{p}\ln\left(\frac{p+\frac{p}{3}}{2\times\frac{p}{3}}\right)$

$\frac{1}{2}\ln 2 = \frac{1}{p}\ln\left(\frac{\frac{4p}{3}}{\frac{2p}{3}}\right)$

$\frac{1}{2}\ln 2 = \frac{1}{p}\ln 2$

$\frac{1}{2} = \frac{1}{p}$

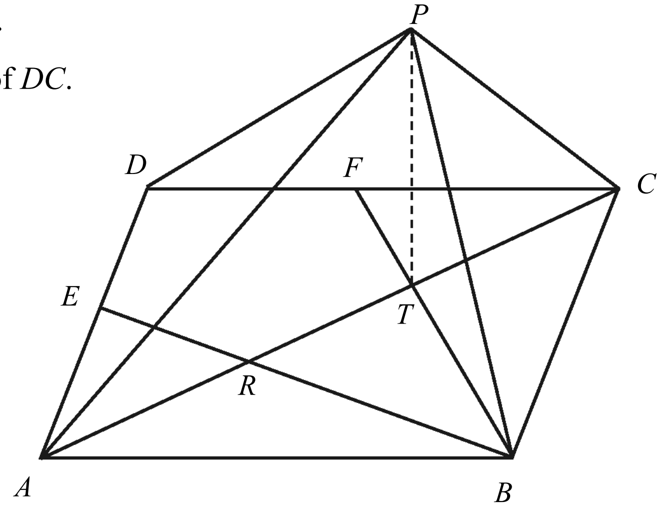
$\therefore\ p = 2$

(b) In the diagram below, $ABCD$ is a parallelogram.

E is the midpoint of AD and F is the midpoint of DC.

R is the point of intersection of AC and BE

T is the point of intersection of AC and BF.



Let $\underset{\sim}{b} = \overrightarrow{AB}$, $\underset{\sim}{c} = \overrightarrow{AC}$, $\underset{\sim}{d} = \overrightarrow{AD}$.

and $ER = kEB$ where k is a scalar.

(i) Show that $\overrightarrow{AR} = k\underset{\sim}{b} + (1-k)\frac{\underset{\sim}{d}}{2}$. **2**

$\overrightarrow{AR} = \overrightarrow{AE} + \overrightarrow{ER}$

$\overrightarrow{ER} = k\overrightarrow{EB}$

$= k(\overrightarrow{AB} - \overrightarrow{AE})$

$= k\left(\overrightarrow{AB} - \frac{\overrightarrow{AD}}{2}\right)$

$= k\left(\underset{\sim}{b} - \frac{\underset{\sim}{d}}{2}\right)$ ✓

$\therefore \overrightarrow{AR} = \overrightarrow{AE} + \overrightarrow{ER}$

$= \frac{\overrightarrow{AD}}{2} + \overrightarrow{ER}$

$= \frac{\underset{\sim}{d}}{2} + k\left(\underset{\sim}{b} - \frac{\underset{\sim}{d}}{2}\right)$

$= k\underset{\sim}{b} + (1-k)\frac{\underset{\sim}{d}}{2}$ ✓

(ii) Hence show that $\overrightarrow{AT} = \dfrac{2}{3}\overrightarrow{AC}$ **3**

ii) $\overrightarrow{AR} = \lambda \overrightarrow{AC}$ for a scalar λ

$= \lambda(\overrightarrow{AB} + \overrightarrow{BC})$

$= \lambda(\overrightarrow{AB} + \overrightarrow{AD})$

$= \lambda(\underset{\sim}{b} + \underset{\sim}{d})$ ✓

from (i) $\overrightarrow{AR} = k\underset{\sim}{b} + (1-k)\dfrac{\underset{\sim}{d}}{2}$

$\lambda(\underset{\sim}{b} + \underset{\sim}{d}) = k\underset{\sim}{b} + (1-k)\dfrac{\underset{\sim}{d}}{2}$

$\lambda = k$, $\lambda = \dfrac{(1-k)}{2}$

$2\lambda = 1-k$

$2k = 1-k$

$3k = 1 \Rightarrow k = \frac{1}{3}$

$\lambda = k = \frac{1}{3}$

$\therefore \overrightarrow{AR} = \frac{1}{3}\overrightarrow{AC}$

Similarly

$\overrightarrow{CT} = \frac{1}{3}\overrightarrow{CA}$

$\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AC}$, $\overrightarrow{CT} = \frac{1}{3}\overrightarrow{CA}$ or $\overrightarrow{TC} = \frac{1}{3}\overrightarrow{AC}$

$\overrightarrow{AR} = \overrightarrow{TC} = \frac{1}{3}\overrightarrow{AC}$

$\therefore \overrightarrow{RT} = \frac{1}{3}\overrightarrow{AC}$

$\overrightarrow{AT} = \overrightarrow{AR} + \overrightarrow{RT}$

$= \frac{1}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AC}$

$= \frac{2}{3}\overrightarrow{AC}$

Hence Shown

The vertices A, B, C and D are joined to a point P in three-dimensional space such that

$\overrightarrow{PT} \bullet \overrightarrow{AC} = 0$

(iii) Prove that $3\left|\overrightarrow{PA}\right|^2 - 3\left|\overrightarrow{PC}\right|^2 = \left|\overrightarrow{AC}\right|^2$ **3**

$$\begin{aligned}
L.H.S. &= 3\left|\overrightarrow{PA}\right|^2 - 3\left|\overrightarrow{PC}\right|^2 \\
&= 3\left(\overrightarrow{PA}\cdot\overrightarrow{PA} - \overrightarrow{PC}\cdot\overrightarrow{PC}\right) \\
&= 3\left((\overrightarrow{PT}+\overrightarrow{TA})\cdot(\overrightarrow{PT}+\overrightarrow{TA}) - (\overrightarrow{PT}+\overrightarrow{TC})\cdot(\overrightarrow{PT}+\overrightarrow{TC})\right) \\
&= 3\left(\left|\overrightarrow{PT}\right|^2 + \left|\overrightarrow{TA}\right|^2 + 2\,\overrightarrow{PT}\cdot\overrightarrow{TA} - \left(\left|\overrightarrow{PT}\right|^2 + \left|\overrightarrow{TC}\right|^2 + 2\overrightarrow{PT}\cdot\overrightarrow{TC}\right)\right) \\
&= 3\left(\left|\overrightarrow{PT}\right|^2 + \left|\overrightarrow{TA}\right|^2 + 0 - \left(\left|\overrightarrow{PT}\right|^2 + \left|\overrightarrow{TC}\right|^2 + 0\right)\right)
\end{aligned}$$

$$\left[\begin{array}{l} \because \overrightarrow{PT} \perp \overrightarrow{TA} \\ \quad\ \overrightarrow{PT} \perp \overrightarrow{TC} \end{array}\right]$$

$$\begin{aligned}
&= 3\left(\cancel{\left|\overrightarrow{PT}\right|^2} + \left|\overrightarrow{TA}\right|^2 - \cancel{\left|\overrightarrow{PT}\right|^2} - \left|\overrightarrow{TC}\right|^2\right) \\
&= 3\left(\left|\overrightarrow{TA}\right|^2 - \left|\overrightarrow{TC}\right|^2\right) \\
&= 3\left(\left|\tfrac{2}{3}\overrightarrow{AC}\right|^2 - \left|\tfrac{1}{3}\overrightarrow{AC}\right|^2\right) \\
&= 3\left(\tfrac{4}{9}\left|\overrightarrow{AC}\right|^2 - \tfrac{1}{9}\left|\overrightarrow{AC}\right|^2\right) \\
&= 3\left(\tfrac{3}{9}\left|\overrightarrow{AC}\right|^2\right) \\
&= 3 \times \tfrac{1}{3}\left|\overrightarrow{AC}\right|^2 \\
&= \left|\overrightarrow{AC}\right|^2 \quad = R.H.S.
\end{aligned}$$

End of Question 15

Alternative Solution:

(iii) $\overrightarrow{PT} \cdot \overrightarrow{AC} = 0$

$\therefore$ $\overrightarrow{PT}$ is perpendicular to $\overrightarrow{AC}$

In ΔPTC

$$|\overrightarrow{PC}|^2 = |\overrightarrow{PT}|^2 + |\overrightarrow{TC}|^2$$

$$|\overrightarrow{PT}|^2 = |\overrightarrow{PC}|^2 - |\overrightarrow{TC}|^2 \quad (1) \quad \checkmark$$

In ΔPAT

$$|\overrightarrow{PA}|^2 = |\overrightarrow{PT}|^2 + |\overrightarrow{AT}|^2$$

$$|\overrightarrow{PA}|^2 = |\overrightarrow{PC}|^2 - |\overrightarrow{TC}|^2 + |\overrightarrow{AT}|^2 \quad \text{Using } (1) \quad \checkmark$$

$$|\overrightarrow{PA}|^2 = |\overrightarrow{PC}|^2 - |\tfrac{1}{3}\overrightarrow{AC}|^2 + |\tfrac{2}{3}\overrightarrow{AC}|^2$$

$$|\overrightarrow{PA}|^2 = |\overrightarrow{PC}|^2 - \frac{1}{9}|\overrightarrow{AC}|^2 + \frac{4}{9}|\overrightarrow{AC}|^2$$

$$|\overrightarrow{PA}|^2 = |\overrightarrow{PC}|^2 + \frac{1}{3}|\overrightarrow{AC}|^2$$

$$3|\overrightarrow{PA}|^2 = 3|\overrightarrow{PC}|^2 + |\overrightarrow{AC}|^2$$

$$\boxed{3|\overrightarrow{PA}|^2 - 3|\overrightarrow{PC}|^2 = |\overrightarrow{AC}|^2} \quad \checkmark \quad \boxed{3}$$

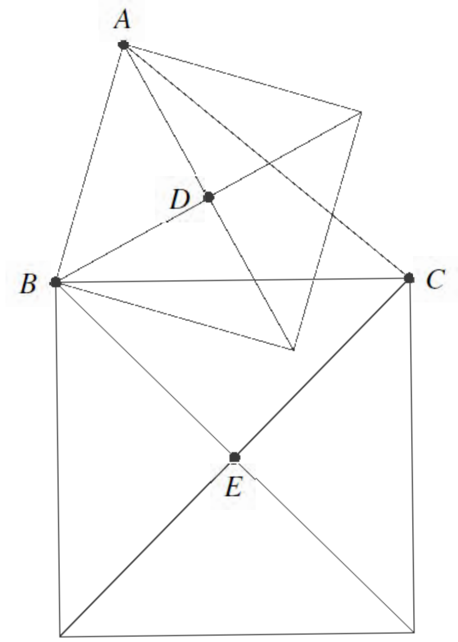
Hence Proved

Question 16. (15 marks) Use a separate writing booklet.

(a) In the diagram below, the complex numbers z_A, z_B, z_C, z_D and z_E correspond to the points *A*, *B*, *C*, *D* and *E* in the complex plane.

AB is the side of a square and *D* is the point of intersection of diagonals of this square.

BC is the side of the larger square and *E* is the point of intersection of diagonals of this square.



(i) Show that $z_B = z_E + (z_C - z_E)i$. **2**

The diagonals of a square meet at 90°

$\therefore \quad \vec{EB} = i\,\vec{EC}$ ✓

$z_B - z_E = i(z_C - z_E)$

$z_B = z_E + (z_C - z_E)i$ ✓

(ii) Hence, or otherwise, show that the angle between the line passing through A and C and the line passing through D and E is $\dfrac{\pi}{4}$ radians **4**

Angle between the line passing through A and C and the line passing through D and E

$= \arg(z_C - z_A) - \arg(z_E - z_D)$ (*)

$z_B = z_E + (z_C - z_E)i$

$z_B = (1-i)z_E + z_C i$

$z_E = \dfrac{z_B - z_C i}{1-i}$ ①

Similarly $\vec{DB} = i\,\vec{DA}$

$z_B - z_D = i(z_A - z_D)$

$z_B = z_D + i(z_A - z_D)$

$z_B = (1-i)z_D + z_A i$

$z_D = \dfrac{z_B - z_A i}{1-i}$ — ②

Using ① and ②

$z_E - z_D = \dfrac{1}{1-i}\left(z_B - z_C i - (z_B - z_A i)\right)$

$= \dfrac{1}{1-i}(\cancel{z_B} - z_C i - \cancel{z_B} + z_A i)$

$= \dfrac{1}{1-i}(z_A - z_C)i$ ③

From (*) and using ③

Required angle

$= \arg(z_C - z_A) - \arg(z_E - z_D)$

$= \arg\left(\dfrac{z_C - z_A}{z_E - z_D}\right)$

$= \arg\left(\dfrac{z_C - z_A}{\frac{1}{1-i}(z_A - z_C)i}\right)$

$= \arg\left(-\dfrac{(1-i)}{i}\right)$

$= \arg\left(-\dfrac{(1-i)}{i} \times \dfrac{i}{i}\right)$

$= \arg\left(-\dfrac{(i+1)}{-1}\right)$

$= \arg(1+i) = \dfrac{\pi}{4}$

Alternative Solution:

$$\overrightarrow{AC} = z_C - z_A$$

$$\overrightarrow{DE} = z_E - z_D$$

$$\overrightarrow{AD} \perp \overrightarrow{BD}$$

$$z_D - z_B = i(z_D - z_A)$$

$$z_B = z_D(1-i) + z_A i$$

$$z_E + (z_C - z_E)i = z_D(1-i) + z_A i \quad [\text{using (i)}]$$

$$z_E + z_C i - z_E i = z_D - z_D i + z_A i$$

$$z_C i - z_E i + z_D i - z_A i = z_D - z_E$$

$$(z_C - z_A)i + (z_D - z_E)i = z_D - z_E$$

$$(z_C - z_A)i = (z_D - z_E)(1-i)$$

$$(z_A - z_C)(-i) = (z_D - z_E)(1-i)$$

$$z_A - z_C = \frac{1-i}{-i}(z_D - z_E)$$

$$z_A - z_C = \frac{1-i}{-i} \times \frac{i}{i}(z_D - z_E)$$

$$z_A - z_C = (1+i)(z_D - z_E)$$

$$z_A - z_C = \sqrt{2}\operatorname{cis}\frac{\pi}{4}(z_D - z_E)$$

$$z_C - z_A = \sqrt{2}\operatorname{cis}\frac{\pi}{4}(z_E - z_D)$$

$$\overrightarrow{AC} = \sqrt{2}\operatorname{cis}\frac{\pi}{4}\,\overrightarrow{DE}$$

$\therefore$ The angle between $\overrightarrow{AC}$ and $\overrightarrow{DE}$ is $\frac{\pi}{4}$

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \, \cos^5\theta \, d\theta$, and $n = 0,1,2,......$

(i) Prove that $I_n = \dfrac{n+1}{n+4} I_{n-1}$ for $n \geq 1$. **3**

Q16(b)(i) 3

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \cos^5\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n+2}\theta \sin\theta \cos^5\theta \, d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \sin^{2n+2}\theta \left(\cos^5\theta(-\sin\theta)\right) d\theta$$

Let $u = \sin^{2n+2}\theta$, $v' = \cos^5\theta(-\sin\theta)$

$u' = (2n+2)\sin^{2n+1}\theta\cos\theta$, $v = \dfrac{\cos^6\theta}{6}$

$$I_n = -\left[\left[\sin^{2n+2}\theta \times \frac{\cos^6\theta}{6}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2n+2)\sin^{2n+1}\theta\cos\theta \times \frac{\cos^6\theta}{6} d\theta\right]$$

$$= -\left[\sin^{2n+2}\frac{\pi}{2} \times \frac{\cos^6\frac{\pi}{2}}{6} - \sin^{2n+2} 0 \times \frac{\cos^6 0}{6}\right]$$

$$+ \frac{n+1}{3}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \cos^7\theta \, d\theta$$

$$= -[0-0] + \frac{n+1}{3}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \times \cos^2\theta \times \cos^5\theta \, d\theta$$

$$= \frac{n+1}{3}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta (1-\sin^2\theta)\cos^5\theta \, d\theta$$

$$= \frac{n+1}{3}\left\{\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\cos^5\theta - \int_0^{\frac{\pi}{2}} \sin^{2n+3}\cos^5\theta \, d\theta\right]$$

$$= \frac{n+1}{3} I_{n-1} - \frac{n+1}{3} I_n$$

Realising that this is I_{n-1}

$$I_n + \left(\frac{n+1}{3}\right) I_n = \frac{n+1}{3} I_{n-1}$$

$$\left(\frac{3+n+1}{3}\right) I_n = \frac{n+1}{3} I_{n-1}$$

$$\left(\frac{n+4}{3}\right) I_n = \left(\frac{n+1}{3}\right) I_{n-1}$$

$$I_n = \frac{n+1}{n+4} I_{n-1}$$

Hence Proved

(ii) Prove that $I_n = \dfrac{1}{(n+4)(n+3)(n+2)}$. **3**

(ii) $I_n = \dfrac{n+1}{n+4} I_{n-1}$

$I_n = \dfrac{n+1}{n+4} \times \dfrac{(n-1)+1}{(n-1)+4} I_{n-2}$

$= \dfrac{n+1}{n+4} \times \dfrac{n}{n+3} I_{n-2}$

$= \dfrac{(n+1)n}{(n+4)(n+3)} \times \dfrac{(n-2)+1}{(n-2)+4} I_{n-3}$

$= \dfrac{(n+1)n(n-1) \dots}{(n+4)(n+3)(n+2)} \times \dfrac{(n-3)+1}{(n-3)+4} I_{n-4}$

$= \dfrac{(n+1)n(n-1) \times (n-2) \times (n-3) \times \dots \times 5 \times 4 \times 3 \times 2}{(n+4)(n+3)(n+2)(n+1) \times n \times \dots \times 8 \times 7 \times 6 \times 5} I_0$ $*$

$I_0 = \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^5\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sin\theta (1-\cos^2\theta) \cos^5\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} (\cos^5\theta - \cos^7\theta) \sin\theta \, d\theta$

$= \left[-\dfrac{\cos^6\theta}{6} + \dfrac{\cos^8\theta}{8} \right]_0^{\frac{\pi}{2}}$

$= \left(-\dfrac{\cos^6\frac{\pi}{2}}{6} + \dfrac{\cos^8\frac{\pi}{2}}{8} \right) - \left(-\dfrac{\cos^6 0}{6} + \dfrac{\cos^8 0}{8} \right)$

$= (-0+0) - \left(-\dfrac{1}{6} + \dfrac{1}{8}\right)$

$= \dfrac{1}{6} - \dfrac{1}{8}$

$= \dfrac{1}{24}$ ✓

$\therefore$ $(*)$ becomes

$I_n = \dfrac{(n+1)n(n-1)(n-2)(n-3) \times \dots \times 5 \times 4 \times 3 \times 2 \times 1}{(n+4)(n+3)(n+2)(n+1)n \dots \times 8 \times 7 \times 6 \times 5} \times \dfrac{1}{24}$

$= \dfrac{(n+1)n(n-1)(n-2)(n-3) \times \dots \times 5 \times 4 \times 3 \times 2 \times 1}{(n+4)(n+3)(n+2)(n+1)n \dots \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= \dfrac{\cancel{(n+1)!}}{(n+4)(n+3)(n+2)\cancel{(n+1)!}}$ $\boxed{3}$

$= \dfrac{1}{(n+4)(n+3)(n+2)}$ Hence Proved ✓

(iii) Let $J_n = \int_0^1 x^{4n+7}\left(1-x^4\right)^2 \, dx$. **3**

Show that $J_n = \frac{1}{2} I_n$.

$$J_n = \int_0^1 x^{4n+7}(1-x^4)^2 \, dx$$

Let $x^2 = \sin\theta$ when $x=0$, $\theta = 0$

$2x\,dx = \cos\theta \, d\theta$ $\quad x=1$, $\theta = \frac{\pi}{2}$

$x\,dx = \frac{\cos\theta}{2} d\theta$

$$J_n = \int_0^1 x^{4n+6} \times x \, (1-x^4)^2 \, dx$$

$$= \int_0^1 (x^2)^{2n+3} \times (1-(x^2)^2)^2 \times x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \times (1-\sin^2\theta)^2 \times \frac{\cos\theta}{2} \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \, (\cos^2\theta)^2 \cos\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \times \cos^4\theta \times \cos\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \, \cos^5\theta \, d\theta$$

$$= \frac{1}{2} I_n$$

3

End of Examination

Fort Street
High School

NESA Number:

Name:

2024 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Approved calculators may be used
- A reference sheet is provided
- Marks may be deducted for careless or badly arranged work.
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks : 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Allow about 2 hours and 45 minutes for this section
- Write your student number on each answer booklet.
- Attempt Questions 11 – 16

Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

---

1. Ler $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Which of the following is the value of $\underset{\sim}{a} \cdot \left(\underset{\sim}{a} - 3\underset{\sim}{b}\right)$?

A. -7

B. 0

C. 3

D. 6

2. Which of the following is an expression for $\displaystyle\int \frac{x^3 - 1}{\left(x^4 - 4x\right)^{\frac{2}{3}}}\,dx$?

A. $\dfrac{3}{4(x^3 - 4x)} + C$

B. $\dfrac{3}{4}(x^3 - 4x) + C$

C. $\dfrac{3}{4}\sqrt[3]{x^4 - 4x} + C$

D. $\dfrac{3}{4\sqrt[3]{x^4 - 4x}} + C$

3. What is the value of $\displaystyle\int_0^{\frac{\pi}{2}} x \sin x \; dx$?

A. $\dfrac{\pi}{4}$

B. $\dfrac{\pi}{2}$

C. π

D. 1

4. In which quadrant does the complex number $2e^{\frac{-i5\pi}{12}} + 2e^{\frac{i\pi}{12}}$ lie ?

A. I

B. II

C. III

D. IV

5. For how many integer values of n ,where $i^2 = -1$, is $n^4 + (n+i)^4$ an integer?

A. 4

B. 3

C. 2

D. 1

6. The complex number $z = a + ib$,where $0 < a < b$.

Which of the following best describes the complex number z^4 ?

A. $\mathrm{Re}(z^4) < 0$

B. $\mathrm{Re}(z^4) \leq 0$

C. $\mathrm{Im}(z^4) < 0$

D. $\mathrm{Im}(z^4) \leq 0$

7. Consider the position vector of a particle.

$$\underset{\sim}{r}(t) = -3\sin(t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j} + t\underset{\sim}{k} \ .$$

Which of the following statements best describes the motion of the particle?

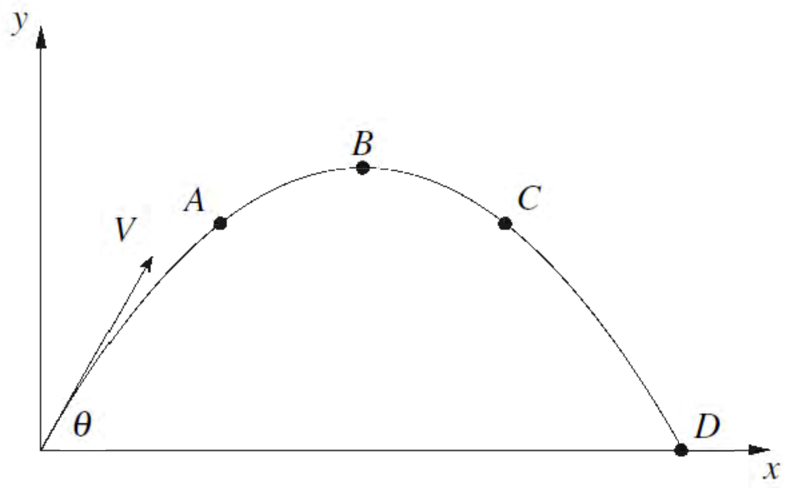
A. A spiral about the z axis in an anticlockwise direction.

B. A spiral about the z axis in a clockwise direction.

C. A spiral about the x axis in an anticlockwise direction.

D. A spiral about the x axis in a clockwise direction.

8. A particle is projected from the origin, reaches a maximum height at point B and lands at point D. The acceleration of the particle is given by $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ and the velocity of the particle is $\underset{\sim}{v}(t)$.



For any initial velocity $V > 0$ and the angle of projection $0 < \theta < 90^\circ$, at which point on the trajectory of the particle are $\underset{\sim}{v}(t) \bullet \underset{\sim}{a}(t) < 0$ and $\underset{\sim}{r}(t) \bullet \underset{\sim}{v}(t) > 0$ always true?

A. A

B. B

C. C

D. D

9. A vehicle is moving horizontally on a frictionless surface in a resistive medium.
The resistive force is proportional to the square of the velocity of the vehicle.
The vehicle has a driving force that varies so that it is always half of the resistive force.
The initial speed of the particle is $5\ ms^{-1}$.

Which of the following is always true about the motion of the particle?

A. The velocity increases until it eventually comes to rest.

B. The velocity decreases until it eventually comes to rest.

C. The velocity increases until it eventually reaches its terminal velocity.

D. The velocity decreases until it eventually reaches $v = 2\ ms^{-1}$

10. The complex number z satisfies $|z+a| = a$,where a is a positive real number.

The point P represents the complex number $ka + a(k+1)i$,
where k is a positive real number.

The greatest distance that z can be at from the point P is $(3\sqrt{2}+1)a$.

What is the value of k ?

A. 1

B. 2

C. 3

D. 4

End of Section I

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet.

(a) The position vectors of two points, A and B are given by $\overrightarrow{OA} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\overrightarrow{OB} = -\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$.

(i) Determine the exact distance between the points A and B. **1**

(ii) Show that there is no value of m such that $\overrightarrow{OC} = m\underset{\sim}{i} + 2\underset{\sim}{j} - m^2\underset{\sim}{k}$ is perpendicular to $\overrightarrow{OA}$. **2**

(b) Solve $z^2 - (2+6i)z = (5-2i)$. Give your answer in the form $x + iy$, where x and y are real. **3**

(c) Find:

$$\int \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$$ **3**

(d) A particle moves along the x-axis with velocity v and acceleration a according to the equation $a = v^3 + 4v$. The particle starts at the origin with velocity $2\ cm/s$.

Find the expression for the displacement of the particle x, in terms of v. **3**

(e) The complex numbers z_1 and z_2 are given by $z_1 = 3 - i$ and $z_2 = 1 - 2i$.

Determine the possible value/s of the real constant k if $\left|\frac{z_1}{z_2} + k\right| = \sqrt{k+2}$. **3**

End of Question 11

Question 12 (15 marks) Use a separate writing booklet.

(a) (i) Find A, B, and C such that **2**

$$\frac{1-2x}{(x+2)(x^2+1)}=\frac{A}{(x+2)}+\frac{Bx+C}{(x^2+1)}$$

(ii) Hence find $\int \frac{1-2x}{(x+2)(x^2+1)}dx$ **2**

(b) Consider the lines **3**

$l_1: \quad x=y=z$

$l_2: \quad \underset{\sim}{r}=\begin{pmatrix}0\\3\\2\end{pmatrix}+t\begin{pmatrix}1\\-2\\1\end{pmatrix}$, where t is a parameter.

Show that the lines are skew.

(c) The polynomial $P(z)=z^4-8z^3+pz^2+qz-80$ has root $3+i$, where p and q are real numbers.

(i) Find all the roots of $P(z)$. **3**

(ii) Write $P(z)$ as a product of two real quadratic factors. **1**

(d) Use the substitution $t=\tan\frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{3}}\frac{1}{4+5\cos x}dx$. **4**

End of Question 12

Question 13 (14 marks) Use a separate writing booklet.

(a) Find $\displaystyle\int \frac{(2\tan\theta+3)\sec^2\theta}{\sec^2\theta+\tan\theta}\,d\theta$ **3**

(b) A particle of mass 1 kg is projected from the origin with initial speed V m/s at an angle α to the horizontal plane.

The parametric equations of motion are given by

$$\ddot{x}=-5\dot{x} \qquad \text{and} \qquad \ddot{y}=-10-5\dot{y}$$

The position vector of the particle , at any time t seconds after the particle is projected , is $\vec{r}(t)$ and the velocity vector is $\vec{v}(t)$.

(i) Show that $\vec{v}(t)=\begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha+2)e^{-5t}-2 \end{pmatrix}$ **3**

(ii) Given that $\vec{v}(1)=\begin{pmatrix} 250e^{-5} \\ \left(250\sqrt{3}+2\right)e^{-5}-2 \end{pmatrix}$, **2**

find the initial speed V and the angle of projection α .

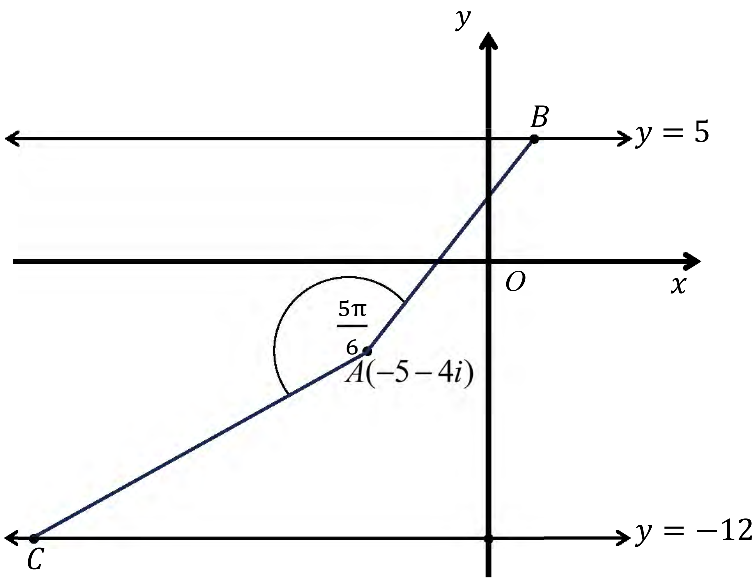
(iii) Show that the ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is: **2**

$$\left(1+125\sqrt{3}\right):1$$

Question 13 continues on page 9

(c) In the Argand diagram below, B lies on the line $y = 5$ and C lies on the line $y = -12$. **4**

The point A represents the complex number $z = -5 - 4i$



NOT TO SCALE

Given $AC = 2AB$ and $\angle BAC = \dfrac{5\pi}{6}$.

Find the exact complex number that represents the point C.

End of Question 13

Question 14. (16 marks) Use a separate writing booklet.

(a) Given that $z = e^{\frac{\pi}{11}i}$, show that

(i) Show that $z + z^3 + z^5 + z^7 + z^9 = \dfrac{1}{1-z}$ **2**

(ii) Hence show that $\cos\dfrac{\pi}{11} + \cos\dfrac{3\pi}{11} + \cos\dfrac{5\pi}{11} + \cos\dfrac{7\pi}{11} + \cos\dfrac{9\pi}{11} = \dfrac{1}{2}$ **3**

(b) (i) Use the substitution $u = \dfrac{1}{x}$ to show that $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = 0$. **2**

(ii) Hence show that $\displaystyle\int_1^3 \frac{\ln x}{3+x^2}\,dx = \frac{\pi\sqrt{3}\ln 3}{36}$. **3**

(c) (i) A complex number z satisfies both $|z-1| \le |z-i|$ and $|z-2-2i| \le 1$. **3**

Sketch on an Argand diagram, the region which contains the point P representing z.

(ii) Point $Q(w)$ lies on the boundary of the region obtained in part (i) **3**

and also satisfies $\arg(w-1) = \dfrac{\pi}{4}$.

Find all possible complex numbers w in the form of $x + iy$ where x and y are real.

End of Question 14

Question 15. (15 marks) Use a separate writing booklet.

(a) A particle of mass m *kg* is moving along the x *axis* under the action of the resisting force given by:

$$m\left(pv+v^2\right),$$

where v is its velocity of particle after t seconds and p is a positive constant.

Initially, the particle is at $x=\ln 2$ and is travelling with velocity p m/s.

(i) Show that the displacement x of the particle is given by **2**

$$x=\ln\left(\frac{4p}{p+v}\right).$$

(ii) Show that: **3**

$$t=\frac{1}{p}\ln\left(\frac{p+v}{2v}\right).$$

(iii) It took the particle $\frac{1}{2}\ln 2$ seconds to reach the point where $x=\ln 3$ meters.

Find the value of p. **2**

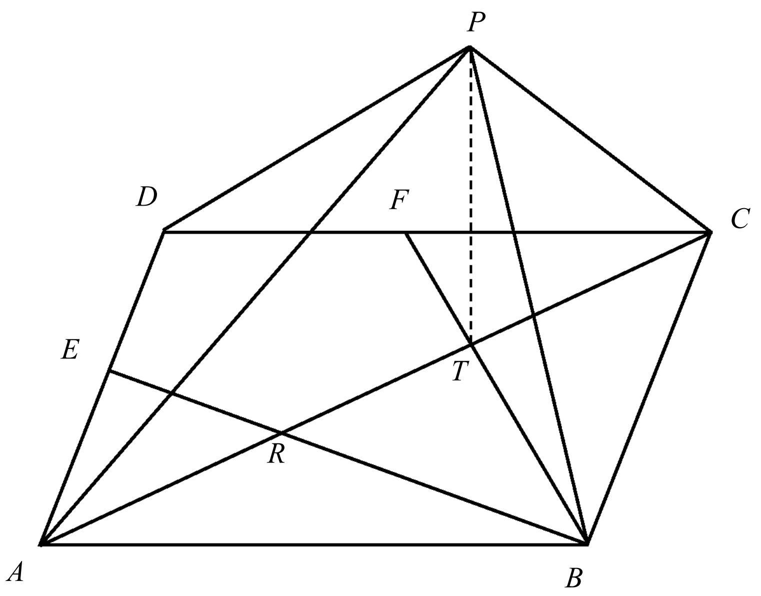
Question 15 continues on page 13

(b) In the diagram below, $ABCD$ is a parallelogram.

E is the midpoint of AD and F is the midpoint of DC.

R is the point of intersection of AC and BE

T is the point of intersection of AC and BF.



Let $\underset{\sim}{b} = \overrightarrow{AB}$, $\underset{\sim}{c} = \overrightarrow{AC}$, $\underset{\sim}{d} = \overrightarrow{AD}$.

and $ER = kEB$ where k is a scalar.

(i) Show that $\overrightarrow{AR} = k\underset{\sim}{b} + (1-k)\dfrac{\underset{\sim}{d}}{2}$. **2**

(ii) Hence show that $\overrightarrow{AT} = \dfrac{2}{3}\overrightarrow{AC}$ **3**

The vertices A, B, C and D are joined to a point P in three-dimensional space such that

$\overrightarrow{PT} \bullet \overrightarrow{AC} = 0$

(iii) Prove that **3**

$$3\left|\overrightarrow{PA}\right|^2 - 3\left|\overrightarrow{PC}\right|^2 = \left|\overrightarrow{AC}\right|^2$$

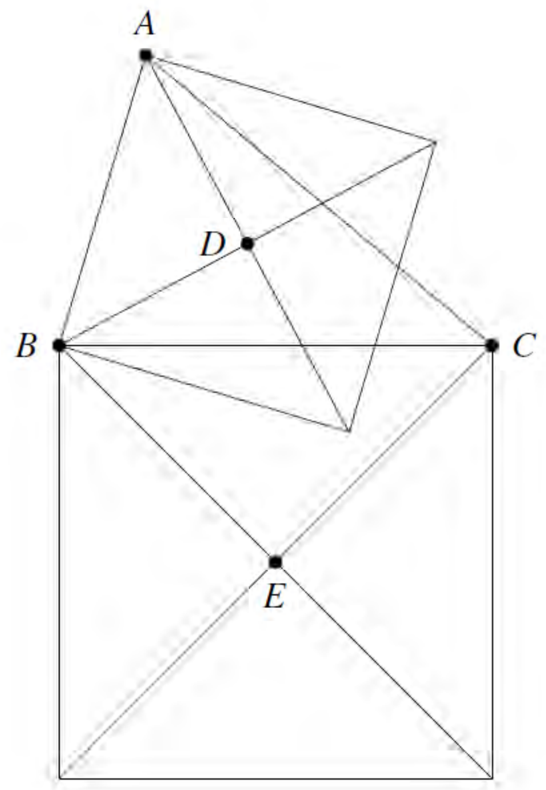
End of Question 15

Question 16. (15 marks) Use a separate writing booklet.

(a) In the diagram below, the complex numbers z_A, z_B, z_C, z_D and z_E correspond to the points *A*, *B*, *C*, *D* and *E* in the complex plane.

AB is the side of a square and D is the point of intersection of diagonals of this square.

BC is the side of the larger square and E is the point of intersection of diagonals of this square.



(i) Show that $z_B = z_E + \left(z_C - z_E\right)i$. **2**

(ii) Hence, or otherwise, show that the angle between the line passing through *A* and *C* and the line passing through D and E is $\frac{\pi}{4}$ radians **4**

Question 16 continues on page 15

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(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+3}\theta \ \cos^5\theta \ \ d\theta$, and $n = 0,1,2,......$

(i) Prove that $I_n = \dfrac{n+1}{n+4} I_{n-1}$ for $n \geq 1$. **3**

(ii) Prove that $I_n = \dfrac{1}{(n+4)(n+3)(n+2)}$. **3**

(iii) Let $J_n = \int_0^1 x^{4n+7}\left(1-x^4\right)^2 \ \ dx$. **3**

Show that $J_n = \dfrac{1}{2} I_n$.

End of Examination